Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Discrete Mathematics I - B2

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Max. points : 40. **Time Limit : 3 hours.**

There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully.

1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points. 1 point will be awarded if only some correct choices are chosen and no wrong choices are chosen.

1. Which of the following statements are true ?

   (a) There exists a graph with one vertex of degree 7 and all other vertices of degree 6.

   (b) A graph on 9 vertices with minimum degree 4 is connected.

   (c) If G is a connected graph on 2023 vertices, then its diameter is at most 2000.

   (d) If G is a connected graph on 2411 vertices, then its diameter is at least 24.

2. Which of the following is true about a tree ?

   (a) It has at least two vertices of degree one.

(b) It has $n-1$ edges.

(c) It has a Hamiltonian cycle.

(d) It has a Eulerian circuit.

3. Let $\alpha'(G)$ be the size of the maximum matching of a graph. Which of the following are correct ?

   (a) $\alpha'(G) = 1$ for a star graph.

   (b) $\alpha'(G) = 6$ for the hypercube graph on $\{0,1\}^3$.

   (c) $\alpha'(G) = 3$ for the cycle C_6 on 6 vertices.

   (d) $\alpha'(G) = 3$ for the path P_7 on 7 vertices.

4. Which of the following are true about complete graph on n vertices ?

   (a) It is a bi-partite graph for all $n \geq 1$.

   (b) It has a perfect matching for all $n \geq 1$.

   (c) The diameter is 1.

   (d) It has n^{n-3} spanning trees.

5. Which of the following are true about designs ?

   (a) A $2-(15,3,1)$ design has at least 15 blocks.

   (b) There is a $3-(16,4,1)$ design that is not a $2-(16,4,\lambda)$ design for any $\lambda \in \mathbb{N}$.

   (c) In $2-(15,3,1)$ design, there is a point contained in at least 5 blocks and a point contained in at most 3 blocks.

   (d) All k-subsets of $[n]$ forms a $(k-1)-(\binom{n}{k}, k, n-k+1)$ design.

2 PART B : 30 Points.

Answer any three questions only. All questions carry 10 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly.

1. Let $S = [nm]$. Let $A_1, \ldots, A_m$ be a partition of S into n-sets. Let $B_1, \ldots, B_m$ be another partition of S into n-sets. Show that there is an ordering of $B_1, \ldots, B_n$ such that $A_i \cap B_i \neq \emptyset$.

2. Let $n \leq 2k$ and $A_1, \ldots, A_m$ be a family of subset of $[n]$ such that $A_i \cup A_j \neq [n], i, j$. Show that $m \leq (1 - \frac{k}{n})\binom{n}{k}$.

3. Let e be an edge in K_n, the complete graph on n-vertices. Show that the number of labelled spanning trees in $K_n - e$ is $(n-2)n^{n-3}$.

4. Find the generating function and use the same to find the sequence explicitly in the following two cases.

   (a) $a_{n+1} = 2a_n + n, n \geq 1$ and $a_0 = 1$.

   (b) $a_{n+1} = a_{n+1} + a_{n-1}, n \geq 1$ and $a_0 = a_1 = 1$.

5. Let $(\mathcal{P}, \mathcal{B})$ be a $t-(v,k,\lambda)$ design and $I \subset \mathcal{P}$ such that $|I| = i \leq t$. Show that the point set $\mathcal{P} \setminus I$ with blocks $\{B \setminus I : I \subset B \in \mathcal{B}\}$ is a design and determine the parameters.